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$G^*_{2(2)}$-Structures on pseudo-Riemannian manifolds

I. Kath [1]

Institut für Mathematik, Humboldt-Universität Berlin, Sitz: Ziegelstr., 13a 10099 Berlin, Germany

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Abstract

We will give the definition and basic properties of nearly parallel $G^*_{2(2)}$-structures on pseudo-Riemannian manifolds of signature (4,3). In particular we explain the equivalence of their existence with that of Killing spinor fields. Furthermore, we will give first examples of pseudo-Riemannian manifolds of signature (4,3) with Killing spinors. © 1998 Elsevier Science B.V.

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1. Introduction

This article relates to the paper of Th. Friedrich et al. [4] on nearly parallel G_2-structures. G_2-structures are topological reductions of the frame bundle of a seven-dimensional manifold to the exceptional group G_2. They can be described by 3-forms of special algebraic type on the manifold. Since $G_2 \subset SO(7)$ such a structure induces a Riemannian metric and in particular a Levi-Civita connection ∇ on the manifold. It is called nearly parallel if the associated 3-form ω^3 satisfies $\nabla_Z \omega^3 = -2\lambda(Z \lrcorner * \omega^3)$. The existence of such a 3-form is equivalent to the existence of a spin structure with a Killing spinor field.

Now we are interested in similar structures on pseudo-Riemannian manifolds, more exactly, on manifolds admitting a metric of signature (4,3). There are two real connected non-compact groups of type G_2. The one with trivial centre denoted by $G^*_{2(2)}$ is a subgroup of $SO(4,3)$. $G^*_{2(2)}$ is one of the possible "exceptional" holonomy groups of non-symmetric irreducible pseudo-Riemannian manifolds [2].

[1] E-mail: kath@mathematik.hu-berlin.de.

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The $Spin(4,3)$-representation $\Delta_{4,3}$ has some algebraic properties similar to those of the $Spin(7)$-representation Δ_7. In particular, both are real. Furthermore, while $Spin(7)$ acts transitively on the sphere S^7 with isotropy group G_2 the action of the connected component $Spin^+(4,3)$ of $Spin(4,3)$ on the pseudo-Riemannian sphere in $\Delta_{4,3}$ is transitive with isotropy group $G^*_{2(2)}$. For a fixed spinor $\psi \neq 0$ in Δ_7 the Clifford multiplication $X \longmapsto X \cdot \psi$ is an isomorphism from $\mathbb{R}^7$ to the orthogonal complement of ψ. The same is true in $\Delta_{4,3}$ for any non-isotropic spinor ψ.

These properties will allow us to translate several results from the Riemannian case to signature (4,3). We will give the definition and basic properties of nearly parallel $G^*_{2(2)}$-structures. In particular, we explain the equivalence of their existence with that of Killing spinor fields. Furthermore, we will give first examples of pseudo-Riemannian spin manifolds of signature (4,3) with Killing spinors.

Analogously to the Riemannian case we have a relation between pairs of Killing spinors and Sasakian structures and between triples of Killing spinors and 3-Sasakian structures on pseudo-Riemannian spin manifolds of signature (4,3). This will be explained in a broader context in [10].

Notation. In the following $\mathbb{R}^{p,q}$ denotes the standard pseudo-Euclidean space of signature (p,q), i.e. $\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, g_{p,q})$ where $g_{p,q}(x,y) = -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{p+q} x_i y_i$. Similarly, $M^{p,q}$ denotes a pseudo-Riemannian manifold of signature (p,q).

2. The exceptional non-compact group $G^*_{2(2)}$

We consider the standard pseudo-Euclidean space $\mathbb{R}^{4,3}$ of signature (4,3) with the standard basis $e_1, e_2, \ldots, e_7$ and define ε_i by $\varepsilon_i = g_{4,3}(e_i, e_i)$. The real Clifford algebra $C_{4,3} = \mathrm{Cliff}(\mathbb{R}^7, -g_{4,3})$ is the algebra generated by $e_1, e_2, \ldots, e_7$ with the relations $e_i^2 = -\varepsilon_i$, $e_i e_j + e_j e_i = 0$ if $i \neq j$. It is isomorphic to the direct sum $\mathbb{R}(8) \oplus \mathbb{R}(8)$ of algebras of real 8×8 matrices. We will use the isomorphism Φ which is defined by

$$
\begin{aligned}
\Phi : C_{4,3} &\longrightarrow \mathbb{R}(8) \oplus \mathbb{R}(8)\\
e_1 &\longmapsto (\varepsilon \otimes \varepsilon \otimes \sigma, \varepsilon \otimes \varepsilon \otimes \sigma)\\
e_2 &\longmapsto (-\sigma \otimes \sigma \otimes \tau, -\sigma \otimes \sigma \otimes \tau)\\
e_3 &\longmapsto (-\sigma \otimes I \otimes \sigma, -\sigma \otimes I \otimes \sigma)\\
e_4 &\longmapsto (\sigma \otimes \tau \otimes \tau, \sigma \otimes \tau \otimes \tau)\\
e_5 &\longmapsto (-I \otimes \varepsilon \otimes \tau, -I \otimes \varepsilon \otimes \tau)\\
e_6 &\longmapsto (-\tau \otimes \varepsilon \otimes \sigma, -\tau \otimes \varepsilon \otimes \sigma)\\
e_7 &\longmapsto (I \otimes I \otimes \varepsilon, -I \otimes I \otimes \varepsilon),
\end{aligned}
\tag{1}
$$

where

$$
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

Usually we will identify $\Phi(e_i)$ with e_i. The projection pr_1 of this isomorphism onto the first component restricted to $Spin(4,3) \subset C_{4,3}$ yields the $Spin(4,3)$-representation on $\mathbb{R}^8 =: \Delta_{4,3}$. Furthermore, this projection defines the Clifford multiplication of a vector $X \in \mathbb{R}^{4,3} \subset C_{4,3}$ with a spinor $\psi \in \Delta_{4,3}$ which we will denote by $X \cdot \psi$. Let u and v be the vectors $u = {}^t(1,0)$, $v = {}^t(0,1)$ and $\psi_1 = u \otimes u \otimes u = {}^t(1,0,\dots,0)$, $\psi_2 = u \otimes u \otimes v = {}^t(0,1,\dots,0), \dots, \psi_8 = v \otimes v \otimes v = {}^t(0,\dots,0,1)$ the standard basis of $\mathbb{R}^8$. We identify the Lie algebra of $Spin(4,3)$ with $\mathfrak{spin}(4,3) = \{\omega = \sum_{i<j} \omega_{ij} e_i e_j \mid \omega_{ij} \in \mathbb{R}\} \subset C_{4,3}$. Let D_{ij} be the 8×8-matrix whose (i,j)-entry is ε_j and all of whose other entries are 0. We set $E_{ij} = -D_{ij} + D_{ji}$. Using this notation $pr_1 \circ \Phi$ becomes with respect to the basis $\psi_1, \dots, \psi_8$

$$\begin{aligned} e_1 &\longmapsto -E_{18} - E_{27} + E_{36} + E_{45},\\ e_2 &\longmapsto E_{17} - E_{28} + E_{35} - E_{46},\\ e_3 &\longmapsto E_{16} + E_{25} + E_{38} + E_{47},\\ e_4 &\longmapsto -E_{15} + E_{26} + E_{37} - E_{48},\\ e_5 &\longmapsto E_{13} - E_{24} - E_{57} + E_{68},\\ e_6 &\longmapsto E_{14} + E_{23} + E_{58} + E_{67},\\ e_7 &\longmapsto -E_{12} - E_{34} + E_{56} + E_{78}. \end{aligned} \tag{2}$$

The following two bilinear forms on $\Delta_{4,3}$ are related to the $Spin(4,3)$-representation. On the one hand we have the standard inner product of $\mathbb{R}^8$ which we denote by $(\ ,\ )$. It is invariant with respect to the maximal compact subgroup $((Pin(4) \times Pin(3))/\mathbb{Z}_2) \cap Spin(4,3)$ of $Spin(4,3)$ and has the property $(X \cdot \varphi, \psi) + (\varphi, \theta(X) \cdot \psi) = 0$ for all $X \in \mathbb{R}^{4,3}$ and $\varphi, \psi \in \Delta_{4,3}$, where $\theta : \mathbb{R}^{4,3} \longrightarrow \mathbb{R}^{4,3}$ denotes the reflection with respect to $\mathrm{span}\{e_5, e_6, e_7\}$. On the other hand we consider the product $\langle\ ,\ \rangle_\Delta$ of signature (4,4) defined by $\langle \varphi, \psi \rangle_\Delta := (e_1 e_2 e_3 e_4 \varphi, \psi)$. It is invariant with respect to the connected component $Spin^+(4,3)$ of $1 \in Spin(4,3)$ and the equation $\langle X \cdot \varphi, \psi \rangle_\Delta + \langle \varphi, X \cdot \psi \rangle_\Delta = 0$ holds for all $X \in \mathbb{R}^{4,3}$ and $\varphi, \psi \in \Delta_{4,3}$. The matrix of $\langle\ ,\ \rangle_\Delta$ with respect to the standard basis $\psi_1, \dots, \psi_8$ equals $\mathrm{diag}(-1,-1,-1,-1,1,1,1,1)$. In particular, we obtain an embedding $Spin(4,3) \subset SO(4,4)$.

Because of the $Spin^+(4,3)$-invariance of $\langle\ ,\ \rangle_\Delta$ the group $Spin^+(4,3)$ acts on $\mathcal{M}_c = \{\psi \in \Delta_{4,3} \mid \langle \psi, \psi \rangle_\Delta = c\}$, $c \in \mathbb{R}$. This action is transitive for $c \neq 0$ and has two orbits for $c = 0$.

Proposition 2.1. *The action of $Spin^+(4,3)$ on*

$$S^{4,3} := \{\psi \in \Delta_{4,3} \mid \langle \psi, \psi \rangle_\Delta = 1\}$$

is transitive. The same is valid for

$$H^{3,4} := \{\psi \in \Delta_{4,3} \mid \langle \psi, \psi \rangle_\Delta = -1\}.$$

The orbits of the $Spin(4,3)^+$-action on

$$\mathcal{C} := \{\psi \in \Delta_{4,3} \mid \langle \psi, \psi \rangle_\Delta = 0\}$$

are $\{0\}$ and $\mathcal{C} \setminus \{0\}$.

Proof. We consider the subspace $\mathbb{R}^{4,1} = \mathrm{span}\{e_1, e_2, e_3, e_4, e_6\}$ of $\mathbb{R}^{4,3}$. The corresponding spin group $Spin^+(4,1) \subset Spin^+(4,3)$ equals $Sp(1,1)$ and $\Delta_{4,3}$ is the standard representation of $Sp(1,1)$ on $\mathbb{R}^{4,4} = \mathbb{H}^{1,1}$. The assertion now follows from the corresponding properties of the $Sp(1,1)$-action on $\mathbb{H}^{1,1}$. □

Corollary 2.1.
1. *The isotropy group $H(\psi) = \{h \in Spin^+(4,3) \mid h\psi = \psi\}$ of a non-isotropic spinor $\psi \in \Delta_{4,3}$ (i.e. $\langle\psi, \psi\rangle_\Delta \neq 0$) with respect to the $Spin^+(4,3)$-action is a connected non-compact group of type G_2 with fundamental group $\mathbb{Z}_2$.*
2. *The Lie algebra of the isotropy group of an isotropic spinor is the semidirect sum of a six-dimensional nilpotent algebra and $\mathfrak{sl}(3,\mathbb{R})$.*

Proof.
1. Because of the transitivity of the $Spin^+(4,3)$-action it suffices to prove that $H(\psi_1)$ has the required properties. We first consider the Lie algebra $\mathfrak{h}(\psi_1)$ of this group. Because of (2) it equals

$$\mathfrak{h}(\psi_1) = \left\{ \sum_{i<j} \omega_n bij e_i e_j \mid -\omega_{12} - \omega_{34} + \omega_{56} = 0, \right.$$
$$\omega_{13} - \omega_{24} - \omega_{67} = 0, -\omega_{14} - \omega_{23} + \omega_{57} = 0,$$
$$\omega_{16} + \omega_{25} - \omega_{37} = 0, \omega_{15} - \omega_{26} - \omega_{47} = 0,$$
$$\left. \omega_{17} + \omega_{36} + \omega_{45} = 0, \omega_{27} + \omega_{35} - \omega_{46} = 0 \right\}. \quad (3)$$

It is spanned by $X_1 = e_1e_2 - e_3e_4$, $Y_1 = e_3e_4 + e_5e_6$, $X_2 = e_1e_3 + e_2e_4$, $Y_2 = e_2e_4 - e_6e_7$, $X_3 = e_1e_4 - e_2e_3$, $Y_3 = e_2e_3 + e_5e_7$, $X_4 = e_1e_6 - e_2e_5$, $Y_4 = e_1e_6 + e_3e_7$, $X_5 = e_2e_6 + e_1e_5$, $Y_5 = e_2e_6 - e_4e_7$, $X_6 = e_1e_7 - e_3e_6$, $Y_6 = e_1e_7 - e_4e_5$, $X_7 = e_2e_7 + e_4e_6$ and $Y_7 = e_2e_7 - e_3e_5$. Using the isomorphism of $\mathfrak{spin}$ (4,3) and $\mathfrak{so}$ (4,3), we see that the Killing form on $\mathfrak{h}(\psi_1)$ is non-degenerate and has index 6. Therefore, $\mathfrak{h}(\psi_1)$ is a non-compact real form of the semisimple Lie algebra $\mathfrak{h}(\psi_1)^{\mathbb{C}}$. Furthermore, one reads from the relations

$$\begin{aligned}
& & [X_1, Y_1] &= 0, \\
[X_1, X_2] &= 4X_3, & [X_1, Y_2] &= 2X_3, \\
[X_1, X_3] &= -4X_2, & [X_1, Y_3] &= 2X_2, \\
[X_1, X_i] &= -2X_{i+1}, & [X_1, Y_i] &= -2Y_{i+1} \quad (i = 4, 6), \\
[X_1, X_j] &= 2X_{j-1}, & [X_1, Y_j] &= 2Y_{j-1} \quad (j = 5, 7), \\
[Y_1, X_2] &= -2X_3, & [Y_1, Y_2] &= 4Y_3, \\
[Y_1, X_3] &= 2X_2, & [Y_1, Y_3] &= -4Y_2,
\end{aligned}$$

that X_1 and Y_1 commute, but no element out of span $\{X_1, Y_1\}$ commutes with both X_1 and Y_1, i.e $\mathfrak{h}(\psi_1)^{\mathbb{C}}$ has rank 2 and thus it must be simple. Since its dimension is 14 it is of type G_2. There is only one non-compact real form of the complex Lie algebra of type G_2 (see e.g. [12]). Now we determine $H(\psi_1)$. Recall that there are two non-compact connected groups of type G_2 (see [12]). The simply connected one has centre $\mathbb{Z}_2$. Because of the transitivity of the $Spin^+(4,3)$-action $H^{3,4}$ is diffeomorphic to the homogeneous space $Spin^+(4,3)/H(\psi_1)$. Using the exact homotopy sequence of this fibration we conclude from $\pi_2(H^{3,4}) = \pi_1(H^{3,4}) = \pi_0(H^{3,4}) = 0$ and from $\pi_1(Spin^+(4,3)) = \mathbb{Z}_2$, $\pi_0(Spin^+(4,3)) = 0$ that $H(\psi_1)$ is connected and has fundamental group $\pi_1(H(\psi_1)) = \mathbb{Z}_2$.

2. We calculate the Lie algebra $\mathfrak{h}(\psi_1 + \psi_5)$ of the isotropy group of $\psi_1 + \psi_5$ and obtain using (2)

$$\begin{aligned} &\mathfrak{h}(\psi_1 + \psi_5) \\ &= \Bigg\{ \sum_{i<j} \omega_{ij} e_i e_j \mid \omega_{16} + \omega_{25} - \omega_{37} = 0, \\ &\qquad \omega_{15} - \omega_{26} - \omega_{12} + \omega_{56} = 0, \omega_{34} - \omega_{47} = 0, \\ &\qquad \omega_{27} + \omega_{23} + \omega_{35} - \omega_{57} = 0, \omega_{14} + \omega_{46} = 0, \\ &\qquad \omega_{13} + \omega_{17} + \omega_{36} - \omega_{67} = 0, \omega_{24} + \omega_{45} = 0 \Bigg\}. \end{aligned} \tag{4}$$

Hence, $\mathfrak{h}(\psi_1 + \psi_5)$ is the semidirect sum of the null space $\mathfrak{n}$ of its Killing form spanned by $e_3e_4 + e_4e_7$, $e_2e_4 - e_4e_5$, $e_1e_4 - e_4e_6$, $e_6e_7 - e_1e_3 + e_1e_7 + e_3e_6$, $e_1e_2 - e_5e_6 + e_1e_5 - e_2e_6$, $e_2e_3 - e_5e_7 - e_2e_7 - e_3e_5$ and the eight-dimensional subalgebra $\mathfrak{p}$ spanned by $e_1e_6 + e_3e_7, e_1e_6 - e_2e_5, e_1e_2 + e_5e_6, e_1e_5 + e_2e_6, e_1e_3 + e_6e_7, e_1e_7 - e_3e_6, e_2e_3 + e_5e_7, e_2e_7 - e_3e_5$. Obviously, $\mathfrak{n}$ is nilpotent. The Killing form restricted to $\mathfrak{p}$ is non-degenerate and has index 3. Consequently, $\mathfrak{p}$ equals $\mathfrak{sl}(3,\mathbb{R})$. □

Definition 2.1. $G^*_{2(2)} := H(\psi_1)$.

Remark. In this notation $H^{3,4}$ is diffeomorphic to $Spin^+(4,3)/G^*_{2(2)}$.

Corollary 2.2. *For a fixed spinor $\psi \in \Delta_{4,3}$ the kernel of the homomorphism*

$$\begin{aligned} \mathbb{R}^{4,3} &\longrightarrow \{\psi\}^\perp \subset \Delta_{4,3} \\ X &\longmapsto X \cdot \psi \end{aligned}$$

(i) *is trivial iff $\psi \neq 0$ is non-isotropic;*
(ii) *has dimension 3 iff $\psi \neq 0$ is isotropic.*

Proof. Using (1) assertions (i) and (ii) can be easily verified for $\psi = \psi_1$ and $\psi = \psi_1 + \psi_5$, respectively. Hence, they hold for any $\psi \neq 0$. □

Now we consider the universal covering $\lambda : Spin(4,3) \longrightarrow SO(4,3)$. Because of $-1 \notin G^*_{2(2)}$ there is an isomorphism from $G^*_{2(2)}$ onto a subgroup of $SO(4,3)$, which we also denote by $G^*_{2(2)}$. We now describe this group using 3-forms on $\mathbb{R}^7$. The key point is a special relation between non-isotropic spinors in $\Delta_{4,3}$ and generic 3-forms in $\Lambda^3(\mathbb{R}^7)$.

We observe that for $X, Y \in \mathbb{R}^{4,3}$ the spinors ψ and $YX\psi + g_{4,3}(X, Y)\psi$ are orthogonal to each other. By Corollary 2.2 we can define a (2,1)-tensor A_ψ by

$$YX\psi + g_{4,3}(X, Y)\psi = A_\psi(Y, X)\psi. \tag{5}$$

A_ψ has the following properties:
(1) $A_\psi(X, Y) = -A_\psi(Y, X)$,
(2) $g_{4,3}(Y, A_\psi(Y, X)) = 0$,
(3) $A_\psi(Y, A_\psi(Y, X)) = -\|Y\|^2_{4,3}X + g_{4,3}(X, Y)Y$.

It defines a 3-form ω^3_ψ by $\omega^3_\psi(X, Y, Z) = g_{4,3}(X, A_\psi(Y, Z))$.

Clearly,

$$\omega^3_{\alpha\psi} = \omega^3_\psi, \quad \alpha \in \mathbb{R}, \ \alpha \neq 0. \tag{6}$$

In particular, if $\psi = \psi_1$ then a direct calculation yields $\omega^3_{\psi_1} = \omega^3_0$, where ω^3_0 is given by

$$\begin{aligned}\omega^3_0 = &-e_1 \wedge e_2 \wedge e_7 - e_1 \wedge e_3 \wedge e_5 + e_1 \wedge e_4 \wedge e_6 \\ &+ e_2 \wedge e_3 \wedge e_6 + e_2 \wedge e_4 \wedge e_5 - e_3 \wedge e_4 \wedge e_7 + e_5 \wedge e_6 \wedge e_7.\end{aligned} \tag{7}$$

Definition 2.2. Let ω^3 be a 3-form on $\mathbb{R}^7$. Furthermore let $X_1, \ldots, X_7$ be an arbitrary pseudo-orthonormal basis of $(\mathbb{R}^7, g_{4,3})$. We define a 4-form σ^4 by $\sigma^4 = \sum_{i=1}^7 \varepsilon_i (X_i \lrcorner \omega^3) \wedge (X_i \lrcorner \omega^3)$ which does not depend on the chosen basis. We will say that ω^3 defines the orientation of $\mathbb{R}^7$ if $\omega^3 \wedge \sigma^4$ is a positive multiple of the volume form of $\mathbb{R}^7$. Furthermore, we will say that ω^3 defines the space and time orientation of $(\mathbb{R}^7, g_{4,3})$ if it defines the orientation of $\mathbb{R}^7$ and if $\omega^3(X_5, X_6, X_7) > 0$ for any positively space and time oriented pseudo-orthonormal basis $X_1, \ldots, X_7$.

Theorem 2.1. *There is a one–one correspondence between $H^{3,4}/\{1, -1\}$ and those $\omega^3 \in \Lambda^3(\mathbb{R}^7)$ which define the space and time orientation of $(\mathbb{R}^7, g_{4,3})$ and for which the bilinear map A defined by $\omega^3(X, Y, Z) = g_{4,3}(X, A(Y, Z))$ has properties* (1)–(3).

Analogously, there is a one – one correspondence between $S^{4,3}/\{1, -1\}$ and those $\omega^3 \in \Lambda^3(\mathbb{R}^7)$ which define the inverse space and time orientation of $(\mathbb{R}^7, g_{4,3})$ and for which the bilinear map A defined by $\omega^3(X, Y, Z) = g_{4,3}(X, A(Y, Z))$ has properties (1)–(3).

Proof. Let $\psi \neq 0$ be a fixed non-isotropic spinor and ω^3_ψ the associated 3-form. With the same notation as above we obtain $\omega^3_\psi \wedge \sigma^4_\psi = 42 e_1 \wedge \cdots \wedge e_7$. Hence, ω^3_ψ defines the orientation of $\mathbb{R}^7$.

Now fix a spinor ψ with $\langle \psi, \psi \rangle_\Delta = -1$ and let $X_1, \ldots, X_7$ be a positively space and time oriented pseudo-orthonormal basis. From the definition of A_ψ we know that $g_{4,3}(A_\psi(X_5, X_6), A_\psi(X_5, X_6)) = 1$ and therefore $A_\psi(X_5, X_6) \notin \{X_5, X_6, X_7\}^\perp$. Since $A_\psi(X_5, X_6) \perp X_5, X_6$ the vectors $A_\psi(X_5, X_6)$ and X_7 cannot be orthogonal. Hence,

$\omega^3_\psi(X_5, X_6, X_7) \neq 0$. Since on the other hand $\omega^3_{\psi_1}(e_5, e_6, e_7) = 1$ we obtain $\omega^3_\psi(X_5, X_6, X_7) > 0$. Hence ω^3_ψ defines the space and time orientation of $(\mathbb{R}^7, g_{4,3})$.

Vice versa, let A be a (2,1)-tensor on $\mathbb{R}^7$ which has the properties (1)–(3). Then A defines a 3-form $\omega^3 = g_{4,3}(\cdot, A(\cdot,\cdot))$. We can define σ^4 in the same way as above. From properties (1)–(3). we conclude $\omega^3 \wedge \sigma^4 \neq 0$. Suppose that ω^3 defines the orientation of $\mathbb{R}^7$. Furthermore, from properties (1)–(3) we deduce as above that $\omega^3(X_5, X_6, X_7) \neq 0$ for any oriented pseudo-orthonormal basis $X_1, \ldots, X_7$. Suppose that ω^3 defines the space and time orientation of $(\mathbb{R}^7, g_{4,3})$. Consider now the subspace

$$E = \{\psi \in \Delta_{4,3} \mid XY\psi = -g_{4,3}(X, Y)\psi + A(X, Y)\psi\}.$$

Then E is one-dimensional and spanned by a spinor ψ_0 with $\langle\psi_0, \psi_0\rangle_\Delta = -1$. In particular, $\omega^3 = \omega_{\psi_0}$. □

In particular, since we have for $g \in Spin^+(4,3)$

$$\omega^3_{g\psi} = (\lambda(g^{-1}))^*\omega_\psi,$$

we conclude:

Corollary 2.3. *The image of $G^*_{2(2)}$ with respect to $\lambda : Spin(4,3) \longmapsto SO(4,3)$ equals*

$$G^*_{2(2)} = \{A \in SO^+(4,3) \mid A^*\omega_0 = \omega_0\}.$$

Note that $A \in SO(4,3)$ and $A^*\omega_0 = \omega_0$ imply $A \in SO^+(4,3)$ since ω_0 defines a space and time orientation.

On the other hand the equation $A^*\omega^3_0 = \omega^3_0$ for $A \in GL(7)$ implies $A \in SO(4,3)$. The proof is similar to that in the G_2-case (see [2]). Consequently, we obtain

$$G^*_{2(2)} = \{A \in GL(7) \mid A^*\omega^3_0 = \omega^3_0\}.$$

Next we investigate in the same way as above the action of $Spin^+(4,3)$ on some of the manifolds

$$\begin{aligned} V(\delta_1, \ldots, \delta_l) = \{(\varphi_1, \ldots, \varphi_l) \mid\ & \varphi_i \in \Delta_{4,3} (i = 1, \ldots, l), \\ & \langle\varphi_i, \varphi_i\rangle_\Delta = \delta_i (i = 1, \ldots, l), \\ & \langle\varphi_i, \varphi_j\rangle_\Delta = 0 \text{ if } i \neq j \ (i, j = 1, \ldots, l)\}, \end{aligned}$$

where $\delta_i = -1$ for $i = 1, \ldots, k$ $(k \leq l)$ and $\delta_i = 1$ for $i = k+1, \ldots, l$.

Proposition 2.2. *The action of $Spin^+(4,3)$ on $V(-1,-1)$, $V(-1,1)$ and $V(1,1)$ is transitive.*

Proof. Since $e_1e_5 \in Spin(4,3)$ maps $S^{4,3}$ one-to-one onto $H^{3,4}$ and

$$(e_1e_5)Spin^+(4,3)(e_1e_5)^{-1} = (e_1e_5)Spin^+(4,3)(-e_5e_1) = Spin^+(4,3), \tag{8}$$

the situation on $V(-1, -1)$ and $V(1, 1)$ is essentially the same.

We calculate the dimension of the isotropy group $H(\varphi_1,\varphi_2)$ of an arbitrary pair (φ_1,φ_2) with $\langle\varphi_1,\varphi_1\rangle_\Delta=-1$, $\langle\varphi_1,\varphi_2\rangle_\Delta=0$ and $\varphi_2\neq 0$. Clearly (see Proposition 2.1), we may assume $\varphi_1=\psi_1$. Next we shall explain why we can assume furthermore $\varphi_2=x_2\psi_2+x_5\psi_5$. The isotropy group $G^*_{2(2)}$ of ψ_1 contains $SO(3)$ and $SU(2)$ as subgroups. The Lie algebra $\mathfrak{so}(3)\subset\mathfrak{so}(4,4)$ is spanned by $e_3e_4+e_5e_6=2(-E_{34}-E_{56})$, $e_2e_4-e_6e_7=2(E_{24}-E_{57})$, $e_2e_3+e_5e_7=2(-E_{23}+E_{67})$ and $\mathfrak{su}(2)\subset\mathfrak{so}(4,4)$ by $e_1e_2-e_3e_4=2(E_{56}-E_{78})$, $e_1e_3+e_2e_4=-2(E_{57}+E_{68})$, $e_1e_4-e_2e_3=2(E_{58}-E_{67})$. Therefore we can first achieve that $\varphi_2=x_2\psi_2+x_5\psi_5+x_6\psi_6+x_7\psi_7+x_8\psi_8$ using the action of $SO(3)\subset G^*_{2(2)}$ and after that $\varphi_2=x_2\psi_2+x_5\psi_5$ using $SU(2)$.

Thus, let φ_2 be $x_2\psi_2+x_5\psi_5$. Eqs. (2) imply that the Lie algebra $\mathfrak{h}(\psi_1,x_2\psi_2+x_5\psi_5)$ of the isotropy group of $(\psi_1,x_2\psi_2+x_5\psi_5)$ equals

$$
\begin{aligned}
&\mathfrak{h}(\psi_1,x_2\psi_2+x_5\psi_5)\\
&=\Big\{\sum_{i<j}\omega_{ij}e_ie_j \mid -\omega_{12}-\omega_{34}+\omega_{56}=0,\\
&\qquad \omega_{13}-\omega_{24}-\omega_{67}=0,\ \omega_{14}+\omega_{23}-\omega_{57}=0,\\
&\qquad -\omega_{16}-\omega_{25}+\omega_{37}=0,\ \omega_{15}-\omega_{26}-\omega_{47}=0,\\
&\qquad \omega_{17}+\omega_{36}+\omega_{45}=0,\ \omega_{27}+\omega_{35}-\omega_{46}=0,\\
&\qquad x_5\omega_{47}=0,\ x_2\omega_{47}=0\\
&\qquad x_2\omega_{57}-x_5\omega_{45}=0,\ x_2\omega_{67}-x_5\omega_{46}=0,\\
&\qquad x_5\omega_{34}+x_2\omega_{37}=0,\ x_5\omega_{24}+x_2\omega_{27}=0,\\
&\qquad x_5\omega_{14}+x_2\omega_{17}=0\Big\}.
\end{aligned}
\tag{9}
$$

Since not $x_2=x_5=0$ the dimension of the Lie algebra $\mathfrak{h}(\varphi_1,\varphi_2)$ of $H(\varphi_1,\varphi_2)$ equals 8 and the one of the orbit of (φ_1,φ_2) equals 13. Hence, all orbits are open sets and the action of $Spin^+(4,3)$ is transitive. □

Corollary 2.4. *The isotropy group of a pair (φ_1,φ_2) of pseudo-orthonormal spinors with respect to the $Spin^+(4,3)$-action equals*
(1) *$SU(1,2)$ if $(\varphi_1,\varphi_2)\in V(-1,-1)$ or $V(1,1)$,*
(2) *$SL(3,\mathbb{R})$ if $(\varphi_1,\varphi_2)\in V(-1,1)$.*

Proof. The Lie algebra of $H(\psi_1,\psi_2)$ equals

$$
\begin{aligned}
\mathfrak{h}(\psi_1,\psi_2)=\Big\{\sum_{i<j}\omega_{ij}e_ie_j \mid &-\omega_{12}-\omega_{34}+\omega_{56}=0,\\
&\omega_{13}-\omega_{24}=0,\ \omega_{14}+\omega_{23}=0,\ \omega_{16}+\omega_{25}=0,\\
&\omega_{15}-\omega_{26}=0,\ \omega_{36}+\omega_{45}=0,\ \omega_{35}-\omega_{46}=0,
\end{aligned}
$$

$$\left.\omega_{i7} = 0, i = 1, \ldots, 6\right\}. \tag{10}$$

As a subalgebra of $\mathfrak{so}$ (4,4) it is spanned by $E_{34} + E_{78}$, $E_{56} - E_{78}$, $E_{57} + E_{68}$, $E_{58} - E_{67}$, $E_{37} + E_{48}$, $E_{38} - E_{47}$, $E_{35} + E_{46}$, $E_{36} - E_{45}$ and equals therefore $\mathfrak{su}$ (1,2) where $SU(1,2) \subset SU(2,2) \subset SO(4,4)$ is embedded in the usual way. We conclude that the connected component of $H(\psi_1, \psi_2)$ must be $SU(1,2)$. On the other hand $V(-1,-1)$ is simply connected. This follows from the exact homotopy sequence of the fibration $SO^+(2,4) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1,-1)$. Using now the exact homotopy sequence of $H(\psi_1, \psi_2) \longrightarrow Spin^+(4,3) \longrightarrow V(-1,-1)$ we deduce from $\pi_1(V(-1,-1)) = 0$ that $H(\psi_1, \psi_2)$ is connected. Thus $H(\psi_1, \psi_2) = SU(1,2)$. □

Now we turn to the Lie algebra of $H(\psi_1, \psi_5)$. It is equal to

$$\mathfrak{h}(\psi_1, \psi_5) = \left\{ \sum_{i<j} \omega_{ij} e_i e_j \mid -\omega_{16} - \omega_{25} + \omega_{37} = 0, \right.$$
$$\omega_{12} - \omega_{56} = 0, \omega_{13} - \omega_{67} = 0, \omega_{23} - \omega_{57} = 0,$$
$$\omega_{15} - \omega_{26} = 0, \omega_{17} + \omega_{36} = 0, \omega_{27} + \omega_{35} = 0,$$
$$\left. \omega_{i4} = 0, i = 1, 2, 3, \omega_{4j} = 0, j = 5, 6, 7 \right\}. \tag{11}$$

Using the isomorphism of $\mathfrak{spin}$ (4,3) and $\mathfrak{so}$ (4,3), we see that the Killing form on $\mathfrak{h}(\psi_1, \psi_5)$ is non-degenerate and has index 3. Therefore, $\mathfrak{h}(\psi_1, \psi_5)$ is a non-compact real form of the semisimple Lie algebra $\mathfrak{h}(\psi_1, \psi_5)^{\mathbb{C}}$. Since, furthermore, $\mathfrak{h}(\psi_1, \psi_5)^{\mathbb{C}}$ has dimension 8 it must be simple and therefore equal to $\mathfrak{sl}(3,\mathbb{C})$. The index of the Killing form distinguishes the various real forms of $\mathfrak{sl}(3,\mathbb{C})$. We conclude that $\mathfrak{h}(\psi_1, \psi_5)$ equals $\mathfrak{sl}(3,\mathbb{R})$. Next we prove that $H(\psi_1, \psi_5)$ is connected and has fundamental group $\mathbb{Z}_2$ which implies immediately $H(\psi_1, \psi_5) = SL(3,\mathbb{R})$ since the centre of the universal covering of $SL(3,\mathbb{R})$ equals $\mathbb{Z}_2$. Using the exact homotopy sequence of the fibration $SO^+(3,3) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1, 1)$ we see that $\pi_2(V(-1, 1)) = \pi_1(V(-1, 1)) = 0$. A look at the exact homotopy sequence of the fibration $H(\psi_1, \psi_5) \longrightarrow Spin^+(4,3) \longrightarrow V(-1, 1)$ now shows that $\pi_1(H(\psi_1, \psi_5)) = \pi_1(Spin^+(4,3)) = \mathbb{Z}_2$ and $\pi_0(H(\psi_1, \psi_5), 1) = 0$. □

Proposition 2.3. *The action of $Spin^+(4,3)$ on the Stiefel manifolds $V(-1,-1,-1)$, $V(-1,-1,1)$, $V(-1,1,1)$ and $V(1,1,1)$ is transitive.*

Proof. As in the proof of Proposition 2.2 it suffices to consider $V(-1,-1,-1)$ and $V(-1,-1,1)$. Again we calculate the Lie algebras of the corresponding isotropy groups. Let φ_1, φ_2 and φ_3 pseudo-orthonormal spinors with $\langle\varphi_1, \varphi_1\rangle_\Delta = -1$ and $\langle\varphi_2, \varphi_2\rangle_\Delta = -1$. Because of Proposition 2.2 we may assume $\varphi_1 = \psi_1$ and $\varphi_2 = \psi_2$. Again the isotropy group of (ψ_1, ψ_2) contains the same subgroup isomorphically to $SU(2)$ as mentioned in the proof of Proposition 2.2 and the group $SO(2) \subset SO(3)$ acting on span$\{\psi_3, \psi_4\}$. Therefore we may set $\varphi_3 = x_3\psi_3 + x_5\psi_5$. Then the isotropy group of $(\varphi_1, \varphi_2, \varphi_3)$ has the Lie algebra

$$\begin{aligned}
&\mathfrak{h}(\psi_1, \psi_2, x_3\psi_3 + x_5\psi_5) \\
&= \Bigg\{ \sum_{i<j} \omega_{ij} e_i e_j \mid -\omega_{12} - \omega_{34} + \omega_{56} = 0, \\
&\qquad \omega_{13} - \omega_{24} = 0, \omega_{14} + \omega_{23} = 0, \omega_{16} + \omega_{25} = 0, \\
&\qquad \omega_{15} - \omega_{26} = 0, \omega_{36} + \omega_{45} = 0, \omega_{35} - \omega_{46} = 0, \\
&\qquad x_4\omega_{56} - x_5\omega_{45} = 0, x_5\omega_{34} + x_4\omega_{36} = 0, \\
&\qquad x_5\omega_{24} + x_4\omega_{26} = 0, x_5\omega_{14} + x_4\omega_{16} = 0, \\
&\qquad x_5\omega_{46} = 0, x_4\omega_{46} = 0, \\
&\qquad \omega_{i7} = 0, i = 1, \ldots, 6 \Bigg\}.
\end{aligned} \tag{12}$$

Since not $x_3 = x_5 = 0$, the dimension of $\mathfrak{h}(\varphi_1, \varphi_2, \varphi_3)$ equals 3 and the action is transitive. □

Corollary 2.5. *The isotropy group of a triple $(\varphi_1, \varphi_2, \varphi_3)$ of pseudo-orthonormal spinors with respect to the $Spin^+(4,3)$-action equals*
1. *$SU(2)$ if $(\varphi_1, \varphi_2, \varphi_3) \in V(-1, -1, -1)$ or $V(1,1,1)$,*
2. *$SL(2,\mathbb{R})$ if $(\varphi_1, \varphi_2, \varphi_3) \in V(-1,-1,1)$ or $V(-1,1,1)$.*

Proof. The Lie algebra of the isotropy group $H(\psi_1, \psi_2, \psi_3)$ of (ψ_1, ψ_2, ψ_3) equals

$$\begin{aligned}
\mathfrak{h}(\psi_1, \psi_2, \psi_3) = \Bigg\{ \sum_{i<j} \omega_{ij} e_i e_j \mid & \omega_{12} + \omega_{34} = 0, \\
& \omega_{13} - \omega_{24} = 0, \omega_{14} + \omega_{23} = 0, \\
& \omega_{i5} = \omega_{i6} = \omega_{i7} = 0 \Bigg\}.
\end{aligned} \tag{13}$$

As a subalgebra of $\mathfrak{so}(4,4)$ it is spanned by $E_{56} - E_{78}$, $E_{57} + E_{68}$ and $E_{58} - E_{67}$ and equals therefore $\mathfrak{su}(2)$ where $SU(2) \subset SU(2,2) \subset SO(4,4)$ is embedded in the usual way. In particular, the connected component of the unity of $H(\psi_1, \psi_2, \psi_3)$ is isomorphic to $SU(2)$. It remains to prove that $H(\psi_1, \psi_2, \psi_3)$ is connected. A look at the exact homotopy sequence of the fibration $H(\psi_1, \psi_2, \psi_3) \longrightarrow Spin^+(4,3) \longrightarrow V(-1, -1, -1)$ shows that it suffices to prove that the group $\pi_1(V(-1, -1, -1))$ equals $\mathbb{Z}_2$. But this is clear from the exact homotopy sequence of $SO^+(1,4) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1, -1, -1)$.

We now prove the second assertion in the same way. The Lie algebra of the isotropy group $H(\psi_1, \psi_5, \psi_6)$ is

$$\mathfrak{h}(\psi_1, \psi_5, \psi_6) = \left\{ \sum_{i<j} \omega_{ij} e_i e_j \mid \omega_{12} - \omega_{56} = 0, \quad \omega_{16} + \omega_{25} = 0, \omega_{15} - \omega_{26} = 0, \quad \omega_{i3} = \omega_{i4} = \omega_{i7} = 0 \right\}. \tag{14}$$

As a subalgebra of $\mathfrak{so}(4,4)$ it is spanned by $E_{34} + E_{78}$, $E_{37} + E_{48}$ and $E_{38} - E_{47}$ and equals therefore $\mathfrak{su}(1,1)$ where $SU(1,1) \subset SU(2,2) \subset SO(4,4)$ is embedded in the usual way. In particular, the connected component of $H(\psi_1, \psi_2, \psi_5)$ is isomorphic to $SU(2)$ which is on the other hand isomorphic to $SL(2;\mathbb{R})$. To show that $H(\psi_1, \psi_5, \psi_6)$ is connected it suffices to verify that the Stiefel manifold is simply connected. But this follows again from the exact homotopy sequence of the corresponding fibration $SO^+(3,2) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1,1,1)$. □

The rest of this section is devoted to real representations of $G^*_{2(2)}$. Recall that all complex representations of $\mathfrak{g}_{2(2)}$ are of real type [12]. Therefore, the real irreducible representations of the universal covering $\widetilde{G_{2(2)}}$ of $G^*_{2(2)}$ correspond to the real forms of the complex irreducible representations of $\mathfrak{g}_{2(2)}$. On the other hand the fundamental representations of $\widetilde{G_{2(2)}}$, i.e. the standard representation on $\mathbb{R}^7$ and the adjoint representation are in fact representations of $G^*_{2(2)}$. Thus all representations of $\widetilde{G_{2(2)}}$ are representations of $G^*_{2(2)}$. We conclude that the real irreducible representations of $G^*_{2(2)}$ correspond exactly to the complex irreducible representations of $\mathfrak{g}_{2(2)}$. In particular, the dimensions of the irreducible real representations are $1, 7, 14, 27, \ldots$ Furthermore, the decomposition of $\Lambda^p(R^7)$ into irreducible components of the $G^*_{2(2)}$-action is similar to that with respect to the action of the compact group G_2. Denote by $*$ the Hodge-operator of the pseudo-Euclidean space $(\mathbb{R}^7, g_{4,3})$ and let ω^3_0 be the 3-form defined by (7). Then we have:

Proposition 2.4.
1. $R^7 = \Lambda^1(R^7) =: \Lambda^1_7$ *is irreducible.*
2. $\Lambda^2(R^7) = \Lambda^2_7 \oplus \Lambda^2_{14}$, *where*

$$\Lambda^2_7 = \{\alpha^2 \in \Lambda^2 \mid *(\omega^3_0 \wedge \alpha^2) = 2\alpha^2\} = \{X \lrcorner \omega^3_0 \mid X \in R^7\}$$
$$\Lambda^2_{14} = \{\alpha^2 \in \Lambda^2 \mid *(\omega^3_0 \wedge \alpha^2) = -\alpha^2\} = \mathfrak{g}_{2(2)}$$

3. $\Lambda^3(R^7) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, *where*

$$\Lambda^3_1 = \{t\omega^3_0 \mid t \in R^1\},$$
$$\Lambda^3_7 = \{*(\omega^3_0 \wedge \alpha^1) \mid \alpha^1 \in \Lambda^1_7\},$$
$$\Lambda^3_{27} = \{\alpha^3 \in \Lambda^3 \mid \alpha^3 \wedge \omega^3_0 = 0, \alpha^3 \wedge *\omega^3_0 = 0\}.$$

3. Killing spinors

Now let $(M^{4,3}, g_{4,3})$ be a seven-dimensional pseudo-Riemannian spin manifold of signature (4,3) which is space and time oriented. Assume that $M^{4,3}$ admits a spin structure $Q(M^{4,3})$. This is a $Spin^+(4,3)$-reduction of the bundle $R(M^{4,3})$ of all space and time oriented pseudo-orthonormal frames. Then the spinor bundle S of $M^{4,3}$ is the associated bundle $Q(M^{4,3}) \times_{Spin^+(4,3)} \Delta_{4,3}$. Furthermore ∇ denotes the Levi-Civita connection on the tangent bundle $TM^{4,3}$ as well as the induced covariant derivative on S. The pseudo-Euclidean product $\langle\ ,\ \rangle_\Delta$ on $\Delta_{4,3}$ induces a product of signature (4,4) on S.

Definition 3.1. A section $\psi \in \Gamma(S)$ is called Killing spinor if there is a real number $\lambda \neq 0$ such that the differential equation

$$\nabla_X \psi = \lambda X \cdot \psi$$

is satisfied for all vector fields $X \in \mathfrak{X}(M^{4,3})$. We call λ the Killing number of ψ.

The following properties of Killing spinors are well-known [1]. Let $\psi \in \Gamma(S)$ be a Killing spinor on $M^{4,3}$ with Killing number λ. Then $\langle\psi, \psi\rangle_\Delta$ is constant on $M^{4,3}$. Hence, it makes sense to say that a Killing spinor is spacelike, timelike, or isotropic. For the Ricci map $Ric : TM^{4,3} \longrightarrow TM^{4,3}$ of the tangent bundle the equation $Ric(X)\psi = 24\lambda^2 X \cdot \psi$ holds. If ψ is non-isotropic, this means that $M^{4,3}$ is an Einstein manifold of scalar curvature $\tau = 168\lambda^2$. Now let W be the Weyl tensor of $M^{4,3}$. Then $W(X, Y) \cdot \psi = 0$ for all $X, Y \in \mathfrak{X}(M^{4,3})$, where this product is defined in the following way. Let $s_1, s_2, \ldots, s_7$ be a local pseudo-orthonormal frame, $\varepsilon_i = g(s_i, s_i)$ and $W_{ijkl} = W(s_i, s_j, s_k, s_l)$. Then

$$W(s_i, s_j) \cdot \psi = \sum_{k<l} \varepsilon_k \varepsilon_l W_{ijkl} s_k \cdot s_l \cdot \psi.$$

Of course, parallel spinors have the same properties. We now turn to the question of how many Killing spinors can exist on $(M^{4,3}, g_{4,3})$.

Theorem 3.1. *If there exist four orthogonal non-isotropic Killing spinors with the same Killing number on $(M^{4,3}, g_{4,3})$ such that at least three of them have the same causal type then $M^{4,3}$ is conformally flat.*

Proof. Let $\varphi_1, \ldots, \varphi_4$ be four such Killing spinors. Let $\langle\varphi_\alpha, \varphi_\alpha\rangle_\Delta = -1$ for $\alpha = 1, 2, 3$. Because of the transitivity of the $Spin^+(4,3)$-action on $V(-1, -1, -1)$ we may assume that for some local time and space oriented pseudo-orthonormal frame $s_1, \ldots, s_7$ the spinor φ_α equals ψ_α for $\alpha = 1, 2, 3$. Moreover, since the isotropy group of (ψ_1, ψ_2, ψ_3) equals $SU(2)$ acting on $\mathrm{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$ we can assume $\varphi_4 = x_4\psi_4 + x_5\psi_5$ where x_4 and x_5 are real functions. The condition $W(s_i, s_j) \cdot \varphi_\alpha = 0$ $(\alpha = 1, 2, 3)$ implies

$$W_{ij12} + W_{ij34} = 0, \qquad W_{ij13} - W_{ij24} = 0, \qquad W_{ij14} + W_{ij23} = 0$$

and $W_{ijkl} = 0$ for any other k, l. Furthermore, we have

$$0 = W(s_i, s_j) \cdot (x_4\psi_4 + x_5\psi_5) = \sum_{k<l} \varepsilon_k \varepsilon_l W_{ijkl} s_k \cdot s_l \cdot (x_4\psi_4 + x_5\psi_5)$$
$$= x_5\{(-W_{ij12} + W_{ij34})\psi_6 + (W_{ij13} + W_{ij24})\psi_7 + (W_{ij14} + W_{ij23})\psi_8\}.$$

Consequently, in case $x_5 \neq 0$ the Weyl tensor must vanish and we are done. Consider now the case $x_5(m) = 0$ for $m \in M^{4,3}$. If there is any sequence $m_n \in M^{4,3}$ which converges to m and such that $x_5(m_n) \neq 0$ then by continuity of the Weyl tensor we have again $W(m) = 0$. Assume now that $x_5(m) = 0$ on an open set containing m, i.e. $\varphi_4 = \psi_4$. Since $\varphi_1, \dots, \varphi_4$ are Killing spinors we have $\nabla_{s_1}\psi_\alpha = \lambda s_1 \cdot \psi_\alpha (\alpha = 1, \dots, 4)$. We can calculate the covariant derivative using the local connection forms $\omega_{ij} = \varepsilon_i \varepsilon_j g_{4,3}(\nabla s_i, s_j)$ and obtain

$$\nabla_{s_1}\psi_\alpha = \frac{1}{2}\sum_{i<j} \varepsilon_i \varepsilon_j \omega_{ij}(s_1) s_i \cdot s_j \cdot \psi_\alpha = \lambda s_1 \cdot \psi_\alpha \quad (\alpha = 1, \dots, 4).$$

In particular,

$$-\omega_{27}(s_1) - \omega_{35}(s_1) + \omega_{46}(s_1) = 2\lambda,$$
$$-\omega_{27}(s_1) + \omega_{35}(s_1) - \omega_{46}(s_1) = 2\lambda,$$
$$-\omega_{27}(s_1) + \omega_{35}(s_1) + \omega_{46}(s_1) = -2\lambda,$$
$$-\omega_{27}(s_1) - \omega_{35}(s_1) - \omega_{46}(s_1) = -2\lambda,$$

which is impossible if $\lambda \neq 0$. The assertion can be proved similarly if $\langle \varphi_\alpha, \varphi_\alpha \rangle_\Delta = 1$ for $\alpha = 1, 2, 3$. □

3.1. Geometrical and nearly parallel $G^*_{2(2)}$-structures

Let M^7 be a seven-dimensional manifold and $R(M^7)$ the frame bundle of M^7. We define the bundle $\Lambda^3_*(M^7)$ by

$$\Lambda^3_*(M^7) := R(M^7) \times_{GL(7)} \Lambda^3_*(R^7) \subset R(M^7) \times_{GL(7)} \Lambda^3(R^7) = \Lambda^3(M^7),$$

where $\Lambda^3_*(R^7)$ is the open subset $\{A^*\omega^3_0 \mid A \in GL(7)\}$ of $\Lambda^3(R^7)$.

Definition 3.2. A topological $G^*_{2(2)}$-structure (*Spin*$^+$(4,3)-structure) on M^7 is a $G^*_{2(2)}$-reduction (*Spin*$^+$(4,3)-reduction) of the frame bundle $R(M^7)$.

The fact that $G^*_{2(2)}$ is a subset of SO^+ (4,3) and of *Spin*$^+$(4,3) implies that a $G^*_{2(2)}$-structure $P \subset R(M^7)$ on M^7 induces an orientation of M^7 (i.e. $\omega_1 = 0$) , a pseudo-Riemannian metric $g_{4,3}$ of index 4 on M^7 together with a space and time orientation such that the corresponding SO^+ (4,3)-bundle equals $P \times_{G^*_{2(2)}} SO^+(4,3)$ and a spin structure $P \times_{G_2} Spin^+(4,3)$. Furthermore it defines the following timelike spinor $\psi \in \Gamma(S)$ in the real spinor bundle $S = P \times_{G^*_{2(2)}} \Delta_{4,3}$ of M^7. Since $G^*_{2(2)} \subset Spin^+(4,3)$ is the isotropy group of $\psi_1 \in \Delta_{4,3}$ the map $\psi : P \longrightarrow \Delta_{4,3}$; $\psi(p) = \psi_1$ has the property $\psi(pg) = g^{-1}\psi$ for all $g \in G^*_{2(2)}$ and is therefore a section in S. Because of the $G^*_{2(2)}$-invariance of ω_0 the $G^*_{2(2)}$-structure defines in the same way a section ω^3 in $\Lambda^3_*(M^7) = R(M^7) \times_{GL(7)} \Lambda^3_*(R^7) =$

$P_{G_2} \times_{G^*_{2(2)}} \Lambda^3_*(R^7)$ by $\omega^3 : P \longrightarrow \Lambda^3_*(R^7)$; $\omega^3(p) = \omega^3_0$. On the other hand the spinor ψ defines a (2,1)-tensor field $A = A_\psi$ (see Eq. (5)) on M^7 and $\omega^3 = g_{4,3}(\cdot, A(\cdot, \cdot))$ holds.

Vice versa, suppose we are given a 3-form ω^3 in $\Lambda^3_*(M^7)$ then M^7 admits a $G^*_{2(2)}$-structure P consisting of all frames relative to those ω^3 equals ω^3_0. Secondly, given a pseudo-Riemannian metric $g_{4,3}$, a space and time orientation, a $Spin^+(4,3)$-structure and a timelike spinor ψ on M^7 then M^7 admits a $G^*_{2(2)}$-structure P consisting of all frames relative to those ψ equals ψ_0.

Now we turn to geometrical $G^*_{2(2)}$-structures.

Definition 3.3. Let $P \subset R(M^7)$ be a topological $G^*_{2(2)}$-structure on M^7 and $g_{4,3}$ the associated Riemannian metric with Hodge operator $*$. P is said to be geometrical if one of the following equivalent conditions is satisfied.

(i) ∇ reduces to P.

(ii) The holonomy group $Hol(M^7, g)$ of M^7 is contained in $G^*_{2(2)}$.

(iii) The associated 3-form ω^3 is parallel, i.e. $\nabla\omega^3 = 0$.

(iv) $d\omega^3 = 0, \quad d * \omega^3 = 0$.

(v) The associated spinor field ψ is parallel, i.e. $\nabla\psi = 0$.

For a proof of (iii) $\Longleftrightarrow$ (iv) see [3,5,6].

Now we can generalize the condition $\nabla\psi = 0$ and obtain the notion of a nearly parallel $G^*_{2(2)}$-structure.

Definition 3.4. Let $P \subset R(M^7)$ be a topological $G^*_{2(2)}$-structure on M^7 and $g_{4,3}$ the associated Riemannian metric with Hodge operator $*$. P is said to be nearly parallel if one of the following equivalent conditions is satisfied.

(i) The associated spinor ψ is a Killing spinor with Killing number λ.

(ii) The associated tensor A satisfies

$$(\nabla_Z A)(Y, X) = 2\lambda\{g_{4,3}(Y, Z)X - g_{4,3}(X, Z)Y + A(Z, A(Y, X))\}.$$

(iii) The associated 3-form ω^3 satisfies

$$\nabla_Z \omega^3 = -2\lambda(Z \lrcorner * \omega^3).$$

(iv) The associated 3-form ω^3 satisfies

$$d * \omega^3 = 0, \quad d\omega^3 = -8\lambda * \omega^3.$$

For a proof of (iii) $\Longleftrightarrow$ (iv) see [4].

3.2. Examples of homogeneous spaces with Killing spinors

In the following we describe various seven-dimensional spaces with homogeneous pseudo-Riemannian metrics of index 4. One can check directly that they all admit a homogeneous spin structure and using Wang's theorem on invariant connections (see [11]) that there are Killing spinors on them. We obtain Section 3.2.1 – 3.2.3 example in remembering that we know seven-dimensional Riemannian homogeneous examples arising as S^1-fibrations over the twistor spaces of S^4 and $\mathbb{C}P^2$ and constructing analogue S^1-fibrations over the twistor spaces of $\mathbb{R}P^{4,0}$, $S^{2,2}$, $\mathbb{C}P^{1,1} = U(2,1)/(U(1) \times U(1,1))$, $\mathbb{C}P^{2,0} = U(2,1)/(U(2) \times U(1))$ and $SO^+(1,1)$-fibrations over the reflector spaces (see [8]) of $S^{2,2}$ and $SL(3,\mathbb{R})/GL^+(2,\mathbb{R})$. The further examples are also in a certain sense dual spaces of known compact Riemannian ones with Killing spinors, namely $V_{5,2} = SO(5)/SO(3)$ and $SO(5)/SO(3)_{\max}$. All examples can be understood in the context of "T-dual" spaces where we have a method to construct pseudo-Riemannian homogeneous spaces with special curvature properties from compact Riemannian ones. This is described in [9].

3.2.1. *The round and the squashed* (4,3)*-sphere*

The standard pseudo-Riemannian sphere $S^{4,3}$ is space and time oriented and admits a homogeneous spin structure. There is an eight-dimensional space of Killing spinors on $S^{4,3}$ to each of both possible Killing numbers. Each of these spaces is spanned by four timelike and four spacelike Killing spinors (with respect to $\langle\ ,\ \rangle_\Delta$). We can consider the following fibrations of $S^{4,3}$ which are similar to the Hopf fibration of the Riemannian sphere S^7:

$$\begin{aligned}
S^3 = Sp(1) &\longrightarrow S^{4,3} = Sp(1,1)/Sp(1) \\
&\longrightarrow S^{4,0}/\mathbb{Z}_2 = \mathbb{H}P^{1,0} = Sp(1,1)/Sp(1) \times Sp(1), \\
S^{2,1} = Sp(2,\mathbb{R}) &\longrightarrow S^{4,3} = Sp(4,\mathbb{R})/Sp(2,\mathbb{R}) \\
&\longrightarrow S^{2,2} = Sp(4,\mathbb{R})/Sp(2,\mathbb{R}) \times Sp(2,\mathbb{R})\,.
\end{aligned}$$

Now we can squash the fibres of these fibrations with scaling factor $\frac{1}{5}$ to obtain in each of both cases a further Einstein metric on the sphere $S^{4,3}$. Both metrics admit a one-dimensional space of non-isotropic Killing spinors. $S^{4,3}$ can be considered as $U(1)$-fibration over the twistor space of $S^{2,2}$ or $S^{4,0}/Z_2$ or as $\mathbb{R}^*$-fibration over the reflector space of $S^{2,2}$.

3.2.2. *The spaces* $\tilde{N}(1,1)$ *and* $\hat{N}(1,1)$

Consider now the homogeneous space $\tilde{N}(1,1) = SU(2,1)/S^1$ where the embedding of S^1 into $SU(2,1)$ is given by

$$\begin{aligned}
S^1 &\hookrightarrow SU(2,1) \\
z &\longmapsto \mathrm{diag}(z, z, z^{-2})
\end{aligned}$$

We decompose $\mathfrak{su}(2,1)$ into $\mathfrak{su}(2,1) = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ where $\mathfrak{k}$ is the Lie algebra of S^1, $\mathfrak{m}_2$ is the Lie algebra of $SU(2) \subset SU(2,1)$, and $\mathfrak{m}_1$ equals $(\mathfrak{k} \oplus \mathfrak{m}_2)^\perp \subset \mathfrak{su}(2,1)$ with respect to the Killing form B of $\mathfrak{su}(2,1)$. Then

$$-B_t = -B\,|_{\mathfrak{m}_1 \times \mathfrak{m}_1} - 2tB\,|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ induces a left invariant metric g_t on $\tilde{N}(1,1)$. $\tilde{N}(1,1)$ is space and time orientable and admits a spin structure. Again there are two possible choices of t to obtain an Einstein metric. In case $t = 1$ we obtain a metric together with three linearly independent Killing spinors of the same causal type and with the same Killing number. For $t = \frac{1}{5}$ we obtain a further Einstein metric on $\tilde{N}(1,1)$ together with a one-dimensional non-isotropic space of Killing spinors.

Next we consider the homogeneous space $\hat{N}(1,1) = SU(1,2)/S^1$ where the embedding of S^1 into $SU(1,2)$ is given by

$$\begin{aligned} S^1 &\hookrightarrow SU(1,2) \\ z &\longmapsto \operatorname{diag}(z, z, z^{-2}). \end{aligned}$$

We decompose $\mathfrak{su}(1,2)$ into $\mathfrak{su}(1,2) = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ where $\mathfrak{k}$ is the Lie algebra of S^1, $\mathfrak{m}_2$ is the Lie algebra of $SU(1,1) \subset SU(1,2)$, and $\mathfrak{m}_1$ equals $(\mathfrak{k} \oplus \mathfrak{m}_2)^{\perp} \subset \mathfrak{su}(1,2)$ with respect to the Killing form B of $\mathfrak{su}(1,2)$. As above

$$-B_t = -B\,|_{\mathfrak{m}_1 \times \mathfrak{m}_1} - 2tB\,|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ defines a left invariant metric g_t on $\hat{N}(1,1)$. $\hat{N}(1,1)$ is space and time orientable and admits a spin structure. Again we obtain in case $t = 1$ an Einstein metric together with three linearly independent Killing spinors, now of different causal type, with the same Killing number. For $t = \frac{1}{5}$ we obtain a further Einstein metric on $\hat{N}(1,1)$ together with a one-dimensional non-isotropic space of Killing spinors.

$\tilde{N}(1,1)$ and $\hat{N}(1,1)$ are S^1-fibrations over $U(2,1)/U(1)\times U(1)\times U(1)$ which is the twistor space of $\mathbb{C}P^{2,0}$ and simultaneously the twistor space of $\mathbb{C}P^{1,1}$. Furthermore, $\tilde{N}(1,1)$ is a fibration over $\mathbb{C}P^{2,0}$ with fibre $\mathbb{R}P^3$ and $\hat{N}(1,1)$ is a fibration over $\mathbb{C}P^{1,1}$ with fibre $\mathbb{R}P^{2,1}$:

$$\begin{aligned} &\mathbb{R}P^3 = U(2)/U(1) \longrightarrow \tilde{N}(1,1) = SU(2,1)/U(1) \longrightarrow \mathbb{C}P^{2,0} = SU(2,1)/U(2) \\ &\mathbb{R}P^{2,1} = U(1,1)/U(1) \longrightarrow \hat{N}(1,1) = SU(1,2)/U(1) \\ &\qquad\qquad\qquad\qquad\qquad \longrightarrow \mathbb{C}P^{1,1} = SU(1,2)/U(1,1). \end{aligned}$$

The second Einstein metric on $\tilde{N}(1,1)$ arises from the first by squashing the $\mathbb{R}P^3$-fibres over $\mathbb{C}P^{0,2}$ and in case of $\hat{N}(1,1)$ the second Einstein metric arises from the first by squashing the $\mathbb{R}P^{2,1}$-fibres over $\mathbb{C}P^{1,1}$.

3.2.3. The space $\bar{N}(1,1)$

Analogously we treat $\bar{N}(1,1) = SL(3,\mathbb{R})/\mathbb{R}^+$ where the embedding of $\mathbb{R}^+$ into $SL(3,\mathbb{R})$ is given by

$$\begin{aligned} \mathbb{R}^+ &\hookrightarrow SL(3,\mathbb{R}) \\ r &\longmapsto \operatorname{diag}(r, r, r^{-2}). \end{aligned}$$

It can be considered as a fibration with fibre $\mathbb{R}^+$ over the double covering of the reflector space of $SL(3,\mathbb{R})/GL^+(2,\mathbb{R})$ and as fibration over $SL(3,\mathbb{R})/GL^+(2,\mathbb{R})$ itself with fibre $S^{2,1}$. $\bar{N}(1,1)$ admits a homogeneous Einstein metric with three linearly independent Killing spinors of different causal type. We get a second Einstein metric in squashing the fibres over $SL(3,\mathbb{R})/GL^+(2,\mathbb{R})$. This squashed metric admits one non-isotropic Killing spinor.

3.2.4. Stiefel manifolds

Let $SO^+(4,1)$, $SO^+(2,3)$, $SO^+(3,2)$ be the connected components of the isometry groups of $\mathbb{R}^{4,1}$, $\mathbb{R}^{2,3}$ and $(\mathbb{R}^5, g = \mathrm{diag}(-1,-1,1,-1,1))$, respectively. The embeddings

$$SO(3) \hookrightarrow SO^+(4,1)\,,\quad SO^+(2,1) \hookrightarrow SO^+(2,3)\,,\quad SO^+(2,1) \hookrightarrow SO^+(3,2)$$

are given by

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

All these Stiefel manifolds are fibrations over the corresponding Grassmann manifolds. If we stretch the homogeneous standard metric (induced by the Killing form) in direction of the fibres by $\frac{3}{2}$ we obtain an Einstein metric with a two-dimensional space of Killing spinors. In case of $SO^+(2,3)/SO^+(2,1)$ all Killing spinors have the same causal type. In both other cases we find non-isotropic Killing spinors of different causal type.

3.2.5. $SO^+(2,3)/SO^+(2,1)_m$

First we describe the embedding of $SO^+(2,1)$ into $SO^+(2,3)$ which we will use here. We denote by $\mathcal{H}^2(\mathbb{R}^{2,1})$ the harmonic (with respect to the indefinite metric), homogeneous polynoms of degree 2 on $\mathbb{R}^3$. We define an indefinite inner product on $\mathcal{H}^2(\mathbb{R}^{2,1})$ by

$$\langle p_1, p_2 \rangle = \int_{S^2} p_1(\mathrm{i}x, \mathrm{i}y, z) p_2(\mathrm{i}x, \mathrm{i}y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

for $p_1, p_2 \in \mathcal{H}^2(\mathbb{R}^{2,1})$. $SO^+(2,1)$ acts on $\mathcal{H}^2(\mathbb{R}^{2,1})$ by $(A \cdot p)(x,y,z) = p(A \cdot (x,y,z))$ for $A \in SO^+(2,1)$, $p \in \mathcal{H}^2(\mathbb{R}^{2,1})$. The infinitesimal actions corresponding to the one-parametric subgroups

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}, \quad \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \quad \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are $A_1 = y \cdot (\partial/\partial z) + z \cdot (\partial/\partial y)$, $A_2 = z \cdot (\partial/\partial x) + x \cdot (\partial/\partial z)$ and $A_3 = x \cdot (\partial/\partial y) - y \cdot (\partial/\partial x)$, respectively. One proves $A_1, A_2, A_3 \in \mathfrak{so}(2,3)$. Hence, we obtain an embedding of $SO^+(2,1)$ into $SO^+(2,3)$. We denote its image by $SO^+(2,1)_m$. There exists a homogeneous Einstein metric on $SO^+(2,3)/SO^+(2,1)_m$ with a one-dimensional space of non-isotropic Killing spinors.

3.3. Warped products with Killing spinors

3.3.1. Pseudo-Riemannian manifolds of signature (4, 2)

Consider first $\mathbb{R}^{4,2} = \mathrm{Span}\{e_1, \ldots, e_6\} \subset \mathbb{R}^{4,3}$. We may restrict the real *Spin*(4,3)-representation to *Spin*(4,2) and obtain the real spinor representation $\Delta_{4,2}$ of *Spin*(4,2). The connected component $Spin^+(4,2)$ of $1 \in Spin(4,2)$ acts transitively on $S^{4,3}$ and $H^{3,4}$. Actually, the proof of Proposition 2.1 remains valid.

The multiplication of spinors by the volume form of $(\mathbb{R}^6, g_{4,2})$ yields a complex structure on $\Delta_{4,2}$. In fact, let $X_1, \dots, X_6$ be a positively oriented pseudo-orthonormal basis of $(\mathbb{R}^6, g_{4,2})$. Then we define J^Δ by $J^\Delta(\psi) = X_1 \cdots X_6 \psi$. J^Δ does not depend on the choice of the pseudo-orthonormal basis. We have $J^\Delta = -I \otimes I \otimes \varepsilon$ with respect to the standard basis $\psi_1, \dots, \psi_8$. Furthermore, J^Δ has the following properties.

1. $(J^\Delta)^2 = -1$.
2. $X \cdot J^\Delta(\psi) = -J^\Delta(X \cdot \psi)$ for any $X \in \mathbb{R}^{4,2}$.
3. Besides $\langle X \cdot \psi, \psi \rangle_\Delta = 0$ we also have $\langle X \cdot \psi, J^\Delta(\psi) \rangle_\Delta = 0$.

Therefore the map

$$\begin{aligned} \mathbb{R}^{4,2} &\longrightarrow \{\psi, J^\Delta(\psi)\}^\perp \subset \mathbb{R}^{4,4} \\ X &\longmapsto X \cdot \psi \end{aligned}$$

is an isomorphism for any spinor $\psi \in \Delta_{4,2}$ with $\langle \psi, \psi \rangle_\Delta \neq 0$. In particular, we obtain a complex structure J_ψ of $\mathbb{R}^{4,2}$ defined by

$$J_\psi(X) \cdot \psi := J^\Delta(X \cdot \psi) \quad \text{for any } X \in \mathbb{R}^{4,2}.$$

Now let $(F^{4,2}, h)$ be a pseudo-Riemannian manifold of signature (4,2). J^Δ defines a complex structure J^S on the spinor bundle S^F of $F^{4,2}$. We have $\nabla J^S = 0$. Assume now that $F^{4,2}$ admits a Killing spinor $\varphi \neq 0$ with Killing number λ. Then obviously $J^S(\varphi)$ is a Killing spinor with Killing number $-\lambda$. Furthermore, any nowhere vanishing nor isotropic section $\psi \in \Gamma(S^F)$ defines a complex structure J_ψ on $F^{4,2}$. If ψ is a non-isotropic Killing spinor then J_ψ is nearly Kählerian.

Next we discuss examples of such manifolds with Killing spinors.

The flag manifold $\tilde{F}(1,2) = SU(2,1)/U(1) \times U(1)$. Consider the homogeneous space $SU(2,1)/U(1) \times U(1)$ where the embedding of $U(1) \times U(1)$ into $U(2,1)$ is given by

$$\begin{aligned} U(1) \times U(1) &\hookrightarrow SU(2,1) \\ (z_1, z_2) &\longmapsto \mathrm{diag}(z_1, z_2, \bar{z}_1 \bar{z}_2). \end{aligned}$$

We decompose $\mathfrak{su}(2,1)$ into $\mathfrak{su}(2,1) = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{m}$ where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ in $\mathfrak{su}(2,1)$ with respect to the Killing form B of $\mathfrak{su}(2,1)$. Now $-B\,|_{\mathfrak{m}\times\mathfrak{m}}$ induces a $SU(2,1)$-invariant Einstein metric on $SU(2,1)/(U(1) \times U(1))$. $\tilde{F}(1,2)$ admits a spin structure. There exists a one-dimensional space of Killing spinors for each of both possible Killing numbers. $SU(2,1)/(U(1) \times U(1))$ can be considered as the twistor space of $\mathbb{C}P^{2,0}$ as well as the twistor space of $\mathbb{C}P^{1,1}$. Note that the metric considered here is not a Kähler–Einstein one. Those can be obtained from it by rescaling the fibres over $\mathbb{C}P^{2,0}$ or $\mathbb{C}P^{1,1}$.

$GL^+(3,\mathbb{R})/\mathbb{R}^+ \times \mathbb{R}^+ \times SO(2)$. The embedding of $\mathbb{R}^+ \times \mathbb{R}^+ \times SO(2)$ into $GL^+(3,\mathbb{R})$ is given by

$$\begin{aligned} \mathbb{R}^+ \times \mathbb{R}^+ \times SO(2) &\hookrightarrow GL^+(3,\mathbb{R}) \\ (r_1, r_2, A) &\longmapsto \begin{pmatrix} r_1 & 0 \\ 0 & r_2 A \end{pmatrix}. \end{aligned}$$

There exist two homogeneous Einstein metrics on $GL^+(3,\mathbb{R})/\mathbb{R}^+ \times \mathbb{R}^+ \times SO(2)$. As above the one induced by the Killing form of $GL^+(3,\mathbb{R})$ admits a one-dimensional non-isotropic space of Killing spinors for each of both possible Killing numbers. $GL^+(3,\mathbb{R})/\mathbb{R}^+ \times \mathbb{R}^+ \times SO(2)$ is the twistor space of $SL(3,\mathbb{R})/GL^+(2,\mathbb{R})$.

$SO^+(4,1)/U(2)$, $SO^+(2,3)/U(1,1)$. Using

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we define the matrix

$J = \begin{pmatrix} J_0 & 0 \\ 0 & J_0 \end{pmatrix}$. Then $U(2)$ is the subgroup $U(2) = \{A \in SO(4) \mid AJ = JA\}$ of $SO(4)$. Furthermore, we have the embedding

$$\begin{aligned} U(2) \subset SO(4) &\hookrightarrow SO(4,1) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Similarly we have $U(1,1) = \{A \in SO^+(2,2) \mid AJ = JA\} \subset SO^+(2,2)$ and the embedding

$$\begin{aligned} U(1,1) \subset SO^+(2,2) &\hookrightarrow SO^+(2,3) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

We decompose $\mathfrak{so}(4,1)$ and $\mathfrak{so}(2,3)$ into $\mathfrak{so}(4,1) = \mathfrak{u}(2) \oplus \mathfrak{m}_1$ and $\mathfrak{so}(2,3) = \mathfrak{u}(1,1) \oplus \mathfrak{m}_2$, where $\mathfrak{m}_1$ is the orthogonal complement of $\mathfrak{u}(2)$ in $\mathfrak{so}(4,1)$ with respect to the Killing form of $\mathfrak{so}(4,1)$ and $\mathfrak{m}_2$ is the orthogonal complement of $\mathfrak{u}(1,1)$ in $\mathfrak{so}(2,3)$ with respect to the Killing form of $\mathfrak{so}(2,3)$. The restrictions of the negative of the corresponding Killing forms to $\mathfrak{m}_1$ and $\mathfrak{m}_2$ induce invariant metrics on $SO^+(4,1)/U(2)$ and $SO^+(2,3)/U(1,1)$, respectively. These metrics are Einstein metrics and admit exactly one non-isotropic Killing spinor for each of the possible Killing numbers. Both spaces are diffeomorphic to $\mathbb{C}P^{2,1}$. We can think of $SO^+(4,1)/U(2)$ as the twistor space of $S^{4,0}/\mathbb{Z}_2 = SO^+(4,1)/SO(4)$ and of $SO^+(2,3)/U(1,1)$ as the twistor space of the sphere $S^{2,2}$. Note that the Einstein metrics with Killing spinors are not the $U(2,2)$-homogeneous Kähler–Einstein metric on $\mathbb{C}P^{2,1}$. They arise from this Kähler–Einstein metric by rescaling the fibres over $S^{4,0}/\mathbb{Z}_2$ and $S^{2,2}$, respectively. The fibres are spacelike in the first case and timelike in the second case. In particular, $SO^+(4,1)/U(2)$ and $SO^+(2,3)/U(1,1)$ are not isometric.

Spin(2,2). We denote by B the Killing form of $\mathfrak{spin}(2,2)$. Let $\mathfrak{m}_1$ be the Lie algebra of *Spin*(2,1) $\subset$ *Spin*(2,2) and $\mathfrak{m}_2$ its orthogonal complement with respect to B. Then $-B\,|_{\mathfrak{m}_1} -3B\,|_{\mathfrak{m}_2}$ induces a left-invariant Einstein metric on *Spin*(2,2) with a one-dimensional non-isotropic space of Killing spinors for each of both possible Killing numbers.

3.3.2. *Pseudo-Riemannian manifolds of signature* (3,3)

Consider now $\mathbb{R}^6 = \mathrm{Span}\{e_1, e_2, e_3, e_5, e_6, e_7\} \subset \mathbb{R}^7$ with pseudo-Euclidean product $g_{3,3} = g_{4,3}|_{\mathbb{R}^6}$. We may restrict the real *Spin*(4,3)-representation to *Spin*(3,3) and obtain the real spinor representation $\Delta_{3,3}$ of *Spin*(3, 3).

The multiplication of spinors by the volume form of $(\mathbb{R}^6, g_{3,3})$ defines now a map J^Δ on $\Delta_{3,3}$ with $(J^\Delta)^2 = 1$. J^Δ anti-commutes with the Clifford multiplication, i.e. $X \cdot J^\Delta(\psi) = -J^\Delta(X \cdot \psi)$ for any $X \in \mathbb{R}^6$. We have $J^\Delta = -\sigma \otimes \tau \otimes \tau$ with respect to the standard basis $\psi_1, \ldots, \psi_8$. Now let $(F^{3,3}, h)$ be a pseudo-Riemannian manifold of signature (3,3). J^Δ defines a map J^S on the spinor bundle S^F of $F^{3,3}$. We have $\nabla J^S = 0$. Assume now that $F^{3,3}$ admits a Killing spinor $\varphi \neq 0$ with Killing number λ. Then obviously $J^S(\varphi)$ is a Killing spinor with Killing number $-\lambda$.

$U(2,1)/U(1) \times SO^+(1,1) \times U(1)$. The embedding of $U(1) \times SO^+(1,1) \times U(1)$ into $U(2,1)$ is given by

$$\begin{aligned} U(1) \times SO^+(1,1) \times U(1) &\hookrightarrow U(2,1) \\ (z_1, A, z_2) &\longmapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 A \end{pmatrix}. \end{aligned}$$

There exist two homogeneous Einstein metrics on $U(2,1)/U(1) \times SO^+(1,1) \times U(1)$. The one induced by the Killing form of $U(2,1)$ admits a one-dimensional non-isotropic space of Killing spinors for each of both possible Killing numbers. $U(2,1)/U(1) \times SO^+(1,1) \times U(1)$ is the reflector space of $\mathbb{C}P^{1,1}$.

$GL^+(3,\mathbb{R})/\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. The embedding of $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ into $GL^+(3,\mathbb{R})$ is given by

$$\begin{aligned} \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ &\hookrightarrow GL^+(3,\mathbb{R}) \\ (r_1, r_2, r_3) &\longmapsto \mathrm{diag}(r_1, r_2, r_3). \end{aligned}$$

There exist two homogeneous Einstein metrics on $GL^+(3,\mathbb{R})/\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. The one induced by the Killing form of $\mathfrak{gl}(3,\mathbb{R})$ admits a one-dimensional non-isotropic space of Killing spinors for each of both possible Killing numbers. $GL^+(3,\mathbb{R})/\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ is a double covering of the reflector space of $SL(3,\mathbb{R})/GL^+(2,\mathbb{R})$.

$SO^+(2,3)/GL^+(2,\mathbb{R})$.

Here we consider $SO^+(3,2)$ as the connected component of the isometry group of $(\mathbb{R}^5, g_{3,2})$, where now $g_{3,2}$ is given with respect to the standard basis by the diagonal matrix $\mathrm{diag}(-1, 1, -1, 1, -1)$. $GL^+(2)$ is embedded in $SO^+(2,3)$ in the following way:

$$\begin{aligned} GL^+(2,\mathbb{R}) &\hookrightarrow SO^+(3,2) \\ N &\longmapsto A \cdot \begin{pmatrix} N & 0 & 0 \\ 0 & ({}^tN)^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot A^{-1}, \end{aligned}$$

where

$$A = \begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

$SO^+(2,3)/GL^+(2)$ admits two homogeneous Einstein metrics. The one which is induced by the restriction of the negative of the Killing form of $\mathfrak{so}(2,3)$ onto the orthogonal complement of $\mathfrak{gl}(2)$ in $\mathfrak{so}(2,3)$ admits a one-dimensional non-isotropic space of Killing spinors for each of both possible Killing numbers. $SO^+(2,3)/GL^+(2,\mathbb{R})$ is the reflector space of $S^{2,2}$.

Spin(3,1). We denote by B the Killing form of $\mathfrak{spin}(3,1)$. Let $\mathfrak{m}_1$ be the Lie algebra of $Spin(3) \subset Spin(3,1)$ and $\mathfrak{m}_2$ its orthogonal complement with respect to B. Then $-B\ |_{\mathfrak{m}_1}$ $-3B\ |_{\mathfrak{m}_2}$ induces a left-invariant Einstein metric on *Spin*(3,1) with a one-dimensional non-isotropic space of Killing spinors for each of both possible Killing numbers.

3.3.3. *Construction of warped products with Killing spinors*

Let $(F^{4,2}, h)$ be a pseudo-Riemannian spin manifold of signature (4,2) with spin structure Q_F and spinor bundle S_F. Furthermore, let $I = (a, b) \subseteq \mathbb{R}$ be an open interval and $\sigma \in C^\infty(I, (0, \infty))$ be a smooth positive function. We consider the warped product

$$(M^{4,3}, g) := F^{4,2} \times_\sigma I := (F^{4,2} \times I, \sigma(t)h \oplus \mathrm{d}t^2).$$

Denote by $\pi : F^{4,2} \times I \longrightarrow F^{4,2}$ the projection. Let $\tilde{Q}$ be that spin structure of $(M^{4,3}, g)$ whose $Spin(n-1)$-reduction with respect to $\xi = \partial/\partial t$ restricted to any fibre $F^{4,2} \times \{t\}$ yields that spin structure of $(F^{4,2}, \sigma(t)h)$ which is conformally equivalent to the spin structure Q_F of $(F^{4,2}, h)$. The spinor bundle S of $(M^{4,3}, g)$ can be identified with the bundle $\pi^* S_F$ by

$$\begin{aligned} \pi^* S_F &\xrightarrow{\sim} S = \tilde{Q} \times_{Spin(4,3)} \Delta_{4,3} \\ \psi = [q, u(x, t)] &\longmapsto \tilde{\psi} = [\tilde{q}, u(x, t)], \end{aligned}$$

where $\tilde{q}$ denotes that element of $\tilde{Q}_{(x,t)}$ which corresponds to $q \in (Q_F)_x$ relative to the conformal equivalence of Q_F and $\tilde{Q}|_{F^{4,2}\times\{t\}}$. For a section $\psi \in \Gamma(\pi^* S_F)$ we denote by $\psi_t \in \Gamma(S_F)$ the spinor field $\psi_t(x) := \psi(x, t)$. Furthermore, for a vector field X on $F^{4,2}$ let $\tilde{X}$ be the vector field $\tilde{X}(x, t) := \sigma(t)^{-1/2} X(x)$ on $M^{4,3}$. Then the following formulae for the Clifford multiplication and the spinor derivative hold:

$$\tilde{X}(x, t) \cdot \tilde{\psi}(x, t) = \widetilde{X(x) \cdot \psi_t(x)}, \tag{15}$$

$$\xi \cdot \tilde{\psi} = -\widetilde{J^S \psi}, \tag{16}$$

$$\nabla_{\tilde{X}} \tilde{\psi} = \sigma(t)^{-1/2} \widetilde{\nabla_X \psi_t} - \tfrac{1}{4} \sigma^{-1} \sigma' \tilde{X} \cdot \xi \cdot \tilde{\psi}, \tag{17}$$

$$\nabla_\xi \tilde{\psi} = \widetilde{\frac{\partial}{\partial t} \psi}. \tag{18}$$

Theorem 3.2. *Let φ^+ and $\varphi^- := J^S(\varphi^+)$ be Killing spinors on $F^{4,2}$ with Killing numbers λ and $-\lambda$, respectively. We may assume $\lambda > 0$. Denote by ψ^+ and ψ^- the sections $\psi^+(x, t) = \cos(\lambda t)\varphi^+(x) - \sin(\lambda t)\varphi^-(x)$ and $\psi^-(x, t) = \sin(\lambda t)\varphi^+(x) - \cos(\lambda t)\varphi^-(x)$ of $\pi^* S_F$. Then $\widetilde{\psi^+}$ and $\widetilde{\psi^-}$ are Killing spinors on $F^{4,2} \times_{\cos^2(2\lambda t)} (-\pi/4\lambda, \pi/4\lambda)$ with Killing numbers λ and $-\lambda$, respectively.*

Proof. Follows by direct calculations using (15)–(18). □

Now we consider the warped product

$$(M^{4,3}, g) := F^{3,3} \times_\sigma I := (F^{3,3} \times I, \sigma(t)h - \mathrm{d}t^2)$$

using a neutral six-dimensional pseudo-Riemannian manifold. Denote by $\pi : F^{3,3} \times I \longrightarrow F^{3,3}$ the projection. $(M^{4,3}, g)$ admits a spin structure $\tilde{Q}$ such that the *Spin*(3,3)-reduction of $\tilde{Q}$ with respect to $\xi = \partial/\partial t$ restricted to any fibre $F^{3,3} \times \{t\}$ yields that spin structure of $(F^{3,3}, \sigma(t)h)$ which is conformally equivalent to the spin structure Q_F of $(F^{3,3}, h)$. As above the spinor bundle S of $(M^{3,3}, g)$ can be identified with the bundle $\pi^* S_F$ and now the following formulae for the Clifford multiplication and the spinor derivative hold.

$$\tilde{X}(x,t) \cdot \widetilde{\psi}(x,t) = \widetilde{X(x) \cdot \psi_t}(x),$$

$$\xi \cdot \widetilde{\psi} = -\widetilde{J^S \psi},$$

$$\nabla_{\tilde{X}} \widetilde{\psi} = \sigma(t)^{-\frac{1}{2}} \widetilde{\nabla_X \psi_t} + \tfrac{1}{4}\sigma^{-1}\sigma' \tilde{X} \cdot \xi \cdot \widetilde{\psi},$$

$$\nabla_\xi \widetilde{\psi} = \widetilde{\frac{\partial}{\partial t}\psi}.$$

Theorem 3.3. *Let now φ^+ and $\varphi^- := J^S(\varphi^+)$ be Killing spinors on $F^{3,3}$ with Killing numbers λ and $-\lambda$, respectively. We may assume $\lambda > 0$. Denote by ψ^+ and ψ^- the sections $\psi^+(x,t) = \cosh(\lambda t)\varphi^+(x) - \sinh(\lambda t)\varphi^-(x)$ and $\psi^-(x,t) = \sinh(\lambda t)\varphi^+(x) + \cosh(\lambda t)\varphi^-(x)$ of $\pi^* S_F$. Then $\widetilde{\psi^+}$ and $\widetilde{\psi^-}$ are Killing spinors on $F^{4,2} \times_{\cosh^2(2\lambda t)} \mathbb{R}$ with Killing numbers λ and $-\lambda$, respectively.*

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